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Irreducible representations of Hecke algebras in the non-standard basis and subduction coefficients

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Abstract. Irreducible representations of Hecke algebras of the type A_{n-1}, $\mathrm{H}_n(q)$, are discussed in the non-standard basis. Some subduction coefficients, i.e. the transformation coefficients between the standard and non-standard base of $\mathrm{H}_n(q)$, are derived by using the linear equation method.

1. Introduction

Recently, braid groups and their representations have drawn increasing attention from physicists and mathematicians [1–10]. Braid group generators can be constructed by using the spectral-independent $\check{R}$-matrices which are the solutions of the Yang–Baxter equation (YBE) without a spectral parameter. The YBE is an important relation in completely integrable systems, e.g. in conformal field theory and the two-dimensional models in statistical physics, which are exactly solvable. An essential tool in the search for such models is the construction of representations of braid groups. From these models some new link polynomials have been constructed, which can be used to distinguish topologically different knots and links [6]. Classification of knots and links is a long-standing problem in mathematics. Thus, a new relationship between physics and mathematics can be established.

The Hecke algebras $\mathrm{H}_n(q)$ are special group realizations, of which the standard basis has been studied in detail by many mathematicians [1–4]. Their standard generators satisfy the same relations as a set of simple reflections (or adjacent permutations) of the symmetric group S_n does, except that the simple property $b_i^2=1$ is replaced by $b_i^2=b_i(q-q^{-1})+1$. It is well known that $\mathrm{H}_n(q)$ is isomorphic to the group algebra of S_n if q is not a root of unity.

The paper is organized as follows. In section 2 we will briefly review the irreducible representations of the Hecke algebras $\mathrm{H}_n(q)$ in the standard basis which is adapted to the chain $\mathrm{H}_n(q)\supset \mathrm{H}_{n-1}(q)\supset \ldots \supset \mathrm{H}_2(q)$. Then the non-standard basis adapted to the chain $\mathrm{H}_n(q)\supset \mathrm{H}_{n_1}(q)\times \mathrm{H}_{n_2}(q)$ is expanded in terms of the standard ones. The expansion coefficients are called the subduction coefficients (SDCs), or the transformation coefficients between the standard and non-standard bases of $\mathrm{H}_n(q)$. In section 3, we

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will outline a linear equation method for evaluating these SDCs. This method is useful in deriving some analytical expressions for the SDCs as well as for the outer product coefficients (ORCs) for the Hecke algebras. In section 4, the analytical expressions of SDCs for $H_n(q)$ with $n \leqslant 5$ and a non-multiplicity-free example for $n=6$ are tabulated.

2. Hecke algebras and their representations

2.1. Hecke algebras

The main subject of this paper deals with the algebras which are homomorphic images of the braid groups. The braid group B_n can be defined topologically [6] or algebraically by the generators $b_1, b_2, \ldots, b_{n-1}$ and the relations

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \qquad \text{for } n=1,2,\ldots,n-1 \tag{2.1}$$

$$b_i b_j = b_j b_i \qquad \text{for } |i-j| \geqslant 2. \tag{2.2}$$

Let $H_n(q)$ be the Hecke algebra of type A_{n-1} over $C(q)$, the field of rational functions over C. $H_n(q)$ is given by the generators $g_1, g_2, \ldots, g_{n-1}$, which satisfy the same relation as that for a set of simple reflections of S_n. It is well known explicity [2] that $H_n(q)$ is not simple only for q being a primitive k^{th} root of unity with $k=2,3,\ldots,n$ or $q=0$. Thus, whenever $H_n(q)$ is semisimple, all its irreducible representations up to conjugacy are labelled by the Young diagrams with n boxes.

2.2. Irreducible representations of $H_n(q)$ in the standard basis

Let $Y_m^{[\lambda]}$ be the standard Young tableau, and $|Y_m^{[\lambda]}\rangle_q$ be the orthogonal basis vectors, i.e. the basis vectors $|Y_m^{[\lambda]}\rangle_q$ satisfy ${}_q\langle Y_m^{[\lambda]}|Y_{m'}^{[\lambda]}\rangle_q = \delta_{mm'}$, where $[\lambda] \equiv [\lambda_1\lambda_2\ldots\lambda_n]$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ and $\lambda_1+\lambda_2+\cdots+\lambda_n = n$, which stands for a standard Young diagram with n boxes, denotes an irrep of $H_n(q)$, and at the same time the Yamanoucchi basis $[\lambda]m$ operates on the indices $(1,\ldots,n)$, where m can be understood either as the Yamanouchi symbols or the indices of the basis vectors in the so-called decreasing page order of the Yamanouchi symbols [13]. Let $g_i(Y_m^{[\lambda]})$ be the standard Young tableau obtained by interchanging the numbers i and $i+1$ in the standard tableau $Y_m^{[\lambda]}$; if $g_i(Y_m^{[\lambda]})$ is not a standard tableau, one sets the corresponding vector equal to 0. Then the irreducible representation of $H_n(q)$ in the standard basis (i.e. the basis adapted to the chain $H_n(q) \supset H_{n-1}(q) \supset \cdots \supset H_2(q)$) is given by

$$g_i|Y_m^{[\lambda]}\rangle_q = \frac{q^d}{[d]}|Y_m^{[\lambda]}\rangle_q + \left(\frac{[d+1][d-1]}{[d]^2}\right)^{1/2}|g_i(Y_m^{[\lambda]})\rangle_q \tag{2.3}$$

where for a given x

$$[x] = \frac{q^x - q^{-x}}{q-q^{-1}} \tag{2.4}$$

and d is the usual axial distance from i to $i+1$ in the Young tableau $Y_m^{[\lambda]}$. A similar result was first obtained by Wenzl [2, 3]. This representation is unitary when $q^* = q^{-1}$ or $q=1$. In the following we always assume that q is generic, i.e. it is not a root of unity.

2.3. *Hecke algebras in the non-standard basis*

An irrep of $\mathrm{H}_n(q)$ is reducible with respect to its subalgebra $\mathrm{H}_{n_1}(q)\times\mathrm{H}_{n_2}(q)$ with $n_1+n_2=n$. The process of the reduction is the same as S_n and is denoted by

$$[\lambda]\downarrow(\mathrm{H}_{n_1}(q)\times\mathrm{H}_{n_2}(q))=\sum_{\lambda}\{\lambda_1\lambda_2\lambda\}([\lambda_1],[\lambda_2]). \tag{2.5}$$

The orthogonal subduced basis is $\mathrm{H}_n(q)\supset\mathrm{H}_{n_1}(q)\times\mathrm{H}_{n_2}(q)$, which is known as the non-standard basis of $\mathrm{H}_n(q)$ (i.e. the basis adapted to the chain $\mathrm{H}_n(q)\supset\mathrm{H}_{n_1}(q)\times\mathrm{H}_{n_2}(q)$ [12], and is denoted by

$$\left|\begin{matrix}[\lambda]\\ \tau\,\lambda_1\,m_1,\lambda_2\,m_2\end{matrix}\right\rangle_q\equiv\left|[\lambda],\begin{matrix}\tau\,[\lambda_1]\,[\lambda_2]\\ m_1\;\;m_2\end{matrix}\right\rangle_q \tag{2.6}$$

where $\tau=1,2,\ldots,[\lambda_1\lambda_2\lambda\}$ is the multiplicity label, and the set of quantum numbers $(\tau[\lambda_1]m_1[\lambda_2]m_2)$ now serves as its component indices.

Similar to the S_n case, we can define a set of the commutants H_i by

$$H_i=H_{i-1}+H_i' \tag{2.7a}$$

$$H_i'=\sum_{k=1}^{i-1}g_{i-1}g_{i-2}\cdots g_k\cdots g_{i-2}g_{i-1} \tag{2.7b}$$

$$H_{i+1}'=g_i+g_iH_i'g_i. \tag{2.7c}$$

Similar to the symmetric group case, H_n is the CSCO-I (the first type of the complete set of commuting operators) of $\mathrm{H}_n(q)$, while (H_{n_1},H_{s_1}) the CSCO-II (the second type of the complete set of commuting operators) of $\mathrm{H}_{n_1}(q)$ (see [12]), where

$$(H_{n_1},H_{s_1})=(H_{n_1},H_{n_1-1},\ldots,H_2). \tag{2.8}$$

We can expand the non-standard basis of $H_n(q)$ in terms of the standard basis of $\mathrm{H}_n(q)$:

$$\left|[\lambda],\begin{matrix}\tau\,[\lambda_1]\,[\lambda_2]\\ m_1\;\;m_2\end{matrix}\right\rangle_q=\sum_m\left|\begin{matrix}[\lambda]\\ m\end{matrix}\right\rangle_q\left\langle\begin{matrix}[\lambda]\\ m\end{matrix}\right|[\lambda],\begin{matrix}\tau\,[\lambda_1]\,[\lambda_2]\\ m_1\;\;m_2\end{matrix}\right\rangle_q. \tag{2.9}$$

The expansion coefficient is called the $[\lambda]\downarrow[\lambda_1]\times[\lambda_2]$ SDC, or the transformation coefficient between the standard and non-standard bases of $\mathrm{H}_n(q)$. Because the standard and non-standard basis vectors are orthogonal, the SDCs satisfy the unitarity conditions

$$\sum_{\lambda_2 m_2\tau}\left\langle\begin{matrix}[\lambda]\\ m\end{matrix}\right|[\lambda],\begin{matrix}\tau\,[\lambda_1]\,[\lambda_2]\\ m_1\;\;m_2\end{matrix}\right\rangle_q\left\langle\begin{matrix}[\lambda]\\ m'\end{matrix}\right|[\lambda],\begin{matrix}\tau\,[\lambda_1]\,[\lambda_2]\\ m_1\;\;m_2\end{matrix}\right\rangle_q=\delta_{mm'} \tag{2.10a}$$

$$\sum_{m}\left\langle\begin{matrix}[\lambda]\\ m\end{matrix}\right|[\lambda],\begin{matrix}\tau\,[\lambda_1]\,[\lambda_2]\\ m_1\;\;m_2\end{matrix}\right\rangle_q\left\langle\begin{matrix}[\lambda]\\ m\end{matrix}\right|[\lambda]\begin{matrix}\tau'\,[\lambda_1]\,[\lambda_2']\\ m_1\;\;m_2'\end{matrix}\right\rangle_q=\delta_{\lambda_2\lambda_2'}\delta_{m_2m_2'}\delta_{\tau\tau'}. \tag{2.10b}$$

The inverse of (2.10) is

$$\left|\begin{matrix}[\lambda]\\ m\end{matrix}\right\rangle_q=\sum_{\lambda_2 m_2}\left|[\lambda],\begin{matrix}\tau\,[\lambda_1]\,[\lambda_2]\\ m_1\;\;m_2\end{matrix}\right\rangle_q\left\langle\begin{matrix}[\lambda]\\ m\end{matrix}\right|[\lambda],\begin{matrix}\tau\,[\lambda_1]\,[\lambda_2]\\ m_1\;\;m_2\end{matrix}\right\rangle_q. \tag{2.11}$$

Once the SDCS are known, the matrix elements of any element $g \in \mathrm{H}_n(q)$ in the non-standard basis can be derived by using the following equation:

$$
{}_q\left\langle [\lambda], \begin{matrix} \tau' [\lambda_1'] [\lambda_2'] \\ m_1' \; m_2' \end{matrix} \middle| g \middle| [\lambda], \begin{matrix} \tau [\lambda_1] [\lambda_2] \\ m_1 \; m_2 \end{matrix} \right\rangle_q
= \sum_{mm'} {}_q\left\langle \begin{matrix}[\lambda] \\ m'\end{matrix} \middle| g \middle| \begin{matrix}[\lambda] \\ m\end{matrix} \right\rangle_q \left\langle \begin{matrix}[\lambda] \\ m\end{matrix} \middle| [\lambda], \begin{matrix} \tau [\lambda_1] [\lambda_2] \\ m_1 \; m_2 \end{matrix} \right\rangle_q \left\langle \begin{matrix}[\lambda] \\ m'\end{matrix} \middle| [\lambda], \begin{matrix} \tau' [\lambda_1'] [\lambda_2] \\ m_1' \; m_2' \end{matrix} \right\rangle_q . \quad (2.12)
$$

3. Evaluation of the SDCS

The eigenfunction method (EFM) proved to be a powerful tool in the evaluation of the CGCS, SDCS and IDCS (induced coefficients) for symmetric groups [12]. These coefficients result from a diagonalization of the CSCO-II of the symmetric group in appropriate spaces. The advantage of this method is that it can easily be programmed and the calculation accomplished using a computer. However, this method can only be used to calculate these coefficients numerically. In the Hecke algebra case, the representations are q-dependent. One wishes to obtain these coefficients with a general q. Thus the diagonalization process becomes rather cumbersome in practice. In the following, we outline a new procedure for evaluating the SDCS of the Hecke algebras.

First, we assume that $\{g_1, g_2, \ldots, g_{n-1}\}$ is a set of generators of $\mathrm{H}_n(q)$, and $\{g_1, g_2, \ldots, g_{n_1-1}\}$, $\{g_{n_1+1}, \ldots, g_{n-1}\}$ are the generators of $\mathrm{H}_{n_1}(q)$, and $\mathrm{H}_{n_2}(q)$, respectively.

By applying g_i with $i = 1, 2, \ldots, n_1 - 1$ and g_j with $j = n_1 + 1$, $n_1 + 2, \ldots, n-1$ to (2.9), and then multiplying the results from the left with

$$
{}_q\left\langle \begin{matrix}[\lambda] \\ m\end{matrix} \right|
$$

we get two sets of linear equations:

$$
\begin{aligned}
&\left(\frac{q^{d_{1i}}}{[d_{1i}]} - \frac{q^{d_i}}{[d_i]}\right) \left\langle \begin{matrix}[\lambda] \\ m\end{matrix} \middle| [\lambda], \begin{matrix} \tau [\lambda_1] [\lambda_2] \\ m_1 \; m_2 \end{matrix} \right\rangle_q \\
&\quad = \left(\frac{[d_i+1][d_i-1]}{[d_i]^2}\right)^{1/2} \left\langle \begin{matrix}[\lambda] \\ m'\end{matrix} \middle| [\lambda], \begin{matrix} \tau [\lambda_1] [\lambda_2] \\ m_1 \; m_2 \end{matrix} \right\rangle_q \\
&\qquad - \left(\frac{[d_{1i}+1][d_{1i}-1]}{[d_{1i}]^2}\right)^{1/2} \left\langle \begin{matrix}[\lambda] \\ m\end{matrix} \middle| [\lambda], \begin{matrix} \tau [\lambda_1] [\lambda_2] \\ m_1' \; m_2 \end{matrix} \right\rangle_q
\end{aligned} \quad (3.1a)
$$

$$
\begin{aligned}
&\left(\frac{q^{d_{2j}}}{[d_{2j}]} - \frac{q^{d_j}}{[d_j]}\right) \left\langle \begin{matrix}[\lambda] \\ m\end{matrix} \middle| [\lambda], \begin{matrix} \tau [\lambda_1] [\lambda_2] \\ m_1 \; m_2 \end{matrix} \right\rangle_q \\
&\quad = \left(\frac{[d_j+1][d_j-1]}{[d_j]^2}\right)^{1/2} \left\langle \begin{matrix}[\lambda] \\ m''\end{matrix} \middle| [\lambda], \begin{matrix} \tau [\lambda_1] [\lambda_2] \\ m_1 \; m_2 \end{matrix} \right\rangle_q \\
&\qquad - \left(\frac{[d_{2j}+1][d_{2j}-1]}{[d_{2j}]^2}\right)^{1/2} \left\langle \begin{matrix}[\lambda] \\ m\end{matrix} \middle| [\lambda], \begin{matrix} \tau [\lambda_1] [\lambda_2] \\ m_1 \; m_2' \end{matrix} \right\rangle_q
\end{aligned} \quad (3.1b)
$$

where $d_{1i}(d_{2j})$ is the axial distance from i to $i+1$ (j to $j+1$) in the Young tableau $Y^{[\lambda_1]}_{m_1}(Y^{[\lambda_2]}_{m_2})$, and $d_i(d_j)$ is the axial distance from i to $i+1$ (j to $j+1$) in the Young tableau $Y^{[\lambda]}_m$. The Young tableaux $Y^{[\lambda]}_{m'}$, $Y^{[\lambda]}_{m''}$, $Y^{[\lambda_1]}_{m_1'}$ and $Y^{[\lambda_2]}_{m_2'}$ are defined by

$$Y^{[\lambda]}_{m'} = g_i Y^{[\lambda]}_m \qquad Y^{[\lambda]}_{m''} = g_j Y^{[\lambda]}_m$$

$$Y^{[\lambda_1]}_{m_1'} = g_i Y^{[\lambda_1]}_{m_1} \qquad Y^{[\lambda_2]}_{m_2'} = g_j Y^{[\lambda_2]}_{m_2}.$$

We assume that the dimensions of $[\lambda]$, $[\lambda_1]$, and $[\lambda_2]$ are N, N_1, and N_2 respectively, which can be calculated by using Robinson's formula [12]. In the following we discuss the multiplicity-free and non-multiplicity-free cases separately. In the multiplicity-free case, there are NN_1N_2 SDCs. Equations (3.1) give $N_1N_2(N-1)$ of linearly independent equations, which give linear dependent relations among the SDCs. Then, using the unitarity relation from (2.11),

$$\sum_m \left\langle \begin{matrix}[\lambda] \\ m\end{matrix} \middle| [\lambda], \begin{matrix}\tau\,[\lambda_1]\,[\lambda_2] \\ m_1 \;\; m_2\end{matrix} \right\rangle_q^2 = 1 \tag{3.2}$$

we can obtain all the SDCs for the given irreps $[\lambda]$, $[\lambda_1]$, and $[\lambda_2]$ because there are exactly N_1N_2 unitarity conditions for the SDCs. Thus, (3.1) and (3.2) are sufficient for solving the SDCs when the subduction is multiplicity-free. An example will now be given to show how this method works.

Example. Find the SDC$\langle[31]m|[31],[1][3]\rangle_q$, where $[\lambda_1]=[1]$, $[\lambda_2]=[3]$, and $[\lambda]=[31]$. First, we rewrite (2.10) as follows:

$$|[31],1,234\rangle_q = a_1\left|{}^{123}_{4}\right\rangle_q + a_2\left|{}^{124}_{3}\right\rangle_q + a_3\left|{}^{134}_{2}\right\rangle_q \tag{3.3}$$

where a_1, a_2, and a_3 are the corresponding SDCs. Applying g_2 and g_3 to (3.3) gives

$$a_1 = ([2]/[4])^{1/2} a_2 \qquad a_2 = (1/[3])^{1/2} a_3. \tag{3.4}$$

Using the unitarity condition (3.2), we have

$$a_1^2 + a_2^2 + a_3^2 = a_3^2(1 + [2]/[4] + [3]) = 1. \tag{3.5}$$

From (3.4) we obtain

$$a_1 = 1/[3] \qquad a_2 = ([4]/[2][3]^2)^{1/2} \qquad a_3 = ([4]/[2][3])^{1/2}. \tag{3.6}$$

In the multiplicity case, similarly to the multiplicity-free case, (3.1) and (3.2) give NN_1N_2 linearly independent relations for the fixed multiplicity label. These relations are sufficient to solve the SDCs with the fixed multiplicity label. However, the same relations hold for any other multiplicity labels. In order to resolve this multiplicity ambiguity, we can use these relations to derive the SDCs for a fixed multiplicity label. Then, the SDCs with different multiplicity labels are chosen to be orthogonal to each other. In this case the solution to the SDCs is not unique and depends on the phase convention and the symmetry properties of SDCs. In this paper, the phase convention for the Hecke algebra SDCs is chosen to be the same as that for the symmetric groups given in [12] (see (4.4) in the next section).

Table 1. $\left\langle \begin{matrix}[\lambda]\\ m\end{matrix} \middle| [\lambda]; \begin{matrix}\tau[\lambda_1] & [\lambda_2]\\ m_1 & m_2\end{matrix} \right\rangle_q$, $[\lambda_2]=[2],[11]$

$[\lambda]m$ \ λ_2	[2]	[11]
$[\lambda]m'$	$\frac{[d+1]}{[2][d]}$	$-\frac{[d-1]}{[2][d]}$
$[\lambda]m$	$\frac{[d-1]}{[2][d]}$	$\frac{[d+1]}{[2][d]}$

Now we give an example to show how to derive the Hecke algebra SDCs with multiplicity. The first non-trivial case occurs in $[321]\downarrow[21]\times[21]$, which has a multiplicity equal to two. Using (3.1) and (3.2), one can easily obtain the SDCs of the first row and the last row in table 7 (see section 4). According to (2.9), the basis of $[321]\downarrow[21]_1\times[21]_\mu$ can be written as

$$|[32]\gamma[21]_1[21]_\mu\rangle_q=\begin{pmatrix} a_{11} & a_{12}\ldots a_{16}\\ a_{21} & a_{22}\ldots a_{26}\\ \ldots\ldots & \\ a_{41} & a_{42}\ldots a_{46}\end{pmatrix}\begin{pmatrix} \left|\begin{matrix}124\\35\\6\end{matrix}\right\rangle_q\\ \left|\begin{matrix}125\\34\\6\end{matrix}\right\rangle_q\\ \vdots\\ \left|\begin{matrix}126\\35\\4\end{matrix}\right\rangle_q\end{pmatrix} \tag{3.7}$$

Table 2. $[\lambda]=[31]$

$\lambda_1;\lambda_2 m_2$ \ $[\lambda]m$	123 4	124 3	134 2
1,234	$\frac{1}{[3]^2}$	$\frac{[4]}{[2][3]^2}$	$\frac{[4]}{[2][3]}$
$1,\begin{matrix}23\\4\end{matrix}$	$\frac{[4][2]}{[3]^2}$	$-\frac{1}{[2]^2[3]^2}$	$-\frac{1}{[2]^2[3]}$
$1,\begin{matrix}24\\3\end{matrix}$		$\frac{[3]}{[2]^2}$	$-\frac{1}{[2]^2}$

Table 3. $[\lambda]=[22]$

	12 34	13 24
$1,{}^{23}_{4}$	$\frac{1}{[2]^2}$	$\frac{[3]}{[2]^2}$
$1,{}^{24}_{3}$	$\frac{[3]}{[2]^2}$	$-\frac{1}{[2]^2}$

where the SDCs are arranged in the same order as those given in table 7. After applying g_4, and g_5 to (3.7), we have

$$\begin{pmatrix} a_{11} & a_{12}\ldots a_{16} \\ a_{21} & a_{22}\ldots a_{26} \\ \cdots\cdots & \\ a_{41} & a_{42}\ldots a_{46} \end{pmatrix}$$

$$=\begin{pmatrix} a_1 & a_1[3]^{1/2} & a_2([3]/[5])^{1/2} & a_2 & a_3 & a_3[3]^{1/2} \\ a_2([3]/[5])^{1/2} & -a_2/[5]^{1/2} & -a_3[5]^{1/2} & a_3[3]^{1/2} & a_4[3]^{1/2} & -a_4 \\ a'_1 & a'_1[3]^{1/2} & a'_2([3]/[5])^{1/2} & a'_2 & a'_3 & a'_3[3]^{1/2} \\ a'_2([3]/[5])^{1/2} & -a'_2/[5]^{1/2} & -a'_3([5])^{1/2} & a'_3[3]^{1/2} & a'_4[3]^{1/2} & -a'_4 \end{pmatrix} \tag{3.8}$$

and

$$a_1+a_2[4]/[2][5]^{1/2}=-a_3[5]^{1/2}$$

$$a_2=-a_3[4]/[2]-a_4 \tag{3.9a}$$

Table 4. $[\lambda]=[41]$

	1234 5	1235 4	1245 3	1345 2
1, 2345	$\frac{1}{[4]^2}$	$\frac{[5]}{[3][4]^2}$	$\frac{[5]}{[4]!}$	$\frac{[5]}{[2][4]}$
$1,{}^{234}_{5}$	$\frac{[3][5]}{[4]^2}$	$-\frac{1}{[3]^2[4]^2}$	$-\frac{1}{[4]![3]}$	$-\frac{1}{[4]!}$
$1,{}^{235}_{4}$		$\frac{[4][2]}{[3]^2}$	$-\frac{1}{[2]^2[3]^2}$	$-\frac{1}{[3]![2]}$
$1,{}^{245}_{3}$			$\frac{[3]}{[2]^2}$	$-\frac{1}{[2]^2}$
12,345	$\frac{[2]}{[3][4]}$	$\frac{[5][2]}{[4][3]^2}$	$\frac{[5]}{[3]^2}$	
$12,{}^{34}_{5}$	$\frac{[5][2]}{[4][3]}$	$-\frac{[2]}{[4][3]^2}$	$-\frac{1}{[3]^2}$	
$12,{}^{35}_{4}$		$\frac{[4]}{[3][2]}$	$-\frac{1}{[3]}$	

Table 5. $[\lambda]=[32]$

	123 45	124 35	134 25	125 34	135 24
$1,{}^{234}_{5}$	$\frac{1}{[3]^2}$	$\frac{[4]}{[2][3]^2}$	$\frac{[4]}{[3]!}$		
$1,{}^{235}_{4}$	$\frac{[4]}{[2][3]^2}$	$-\frac{1}{[3]^2[2]^4}$	$-\frac{1}{[3][2]^4}$	$\frac{[3]}{[2]^4}$	$\frac{[3]^2}{[2]^4}$
$1,{}^{245}_{3}$		$\frac{[3]}{[2]^4}$	$-\frac{1}{[2]^4}$	$\frac{[3]^2}{[2]^4}$	$-\frac{[3]}{[2]^4}$
$1,{}^{23}_{45}$	$\frac{[4]}{[3]!}$	$-\frac{1}{[3][2]^4}$	$-\frac{1}{[2]^4}$	$-\frac{1}{[2]^4}$	$-\frac{[3]}{[2]^4}$
$1,{}^{24}_{35}$		$\frac{[3]^2}{[2]^4}$	$-\frac{[3]}{[2]^4}$	$-\frac{[3]}{[2]^4}$	$\frac{1}{[2]^4}$
$12,345$	$\frac{1}{[3]^2}$	$\frac{[4]}{[2][3]^2}$		$\frac{[4]}{[3]!}$	
$12,{}^{34}_{5}$	$\frac{[4]}{[2][3]^2}$	$\frac{[4]^2}{([3]!)^2}$		$\frac{1}{[3]}$	
$12,{}^{35}_{4}$	$\frac{[4]}{[3]!}$	$-\frac{1}{[3]}$			

$$a_1'+a_2'[4]/[2][5]^{1/2}=-a_3'[5]^{1/2}$$

$$a_2'=-a_3'[4]/[2]-a_4'. \tag{3.9b}$$

Then, using the unitarity conditions,

$$a_1^2[2]^2+a_2^2[4][2]/[5]+a_3^2[2]^2=1$$

$$a_2^2[2]^2/[5]+a_3^2[4][2]+a_4^2[2]^2=1 \tag{3.10}$$

and the same relations for a_i's, we can establish four independent relations which are sufficient to solve the a_is. Then the SDCs of the third and fourth rows in (3.8) are chosen to be orthogonal to the first and second rows in (3.8). It can be seen that, by the substitutions

$$a_1'\to a_4, \qquad a_2'\to a_3[5]^{1/2}$$

$$a_3'\to a_2/[5]^{1/2} \qquad a_4'\to a_1 \tag{3.11}$$

the new variables also satisfy (3.9*a*) and (3.10). This is nothing but the symmetry condition

$$\left\langle \begin{matrix}[321]\\ \tilde{m}\end{matrix} \middle| \begin{matrix}\alpha\,[21]\,[21]\\ \tilde{m}_1\;\;\tilde{m}_2\end{matrix}\right\rangle_q = \Lambda_m^{[321]}\Lambda_{m_1}^{[21]}\Lambda_{m_2}^{[21]}\left\langle \begin{matrix}[321]\\ m\end{matrix} \middle| \begin{matrix}\beta\,[21]\,[21]\\ m_1\;\;m_2\end{matrix}\right\rangle_q \tag{3.12}$$

where $\Lambda_m^{[\lambda]}$s are the phases of the corresponding Yamanouchi basis, and $[\tilde{\lambda}]\tilde{m}$ denotes a Young tableau conjugate to $[\lambda]m$. The following symmetry is also used in deriving (3.12):

$$\left\langle \begin{matrix}[321]\\ m\end{matrix} \middle| \begin{matrix}\gamma\,[21]\,[21]\\ 2\quad m'\end{matrix} \right\rangle_q = \varepsilon([21][21][321])\Lambda^{[321]}_{\tilde{m}}\Lambda^{[21]}_{1}\Lambda^{[21]}_{\tilde{m}'}\left\langle \begin{matrix}[321]\\ \tilde{m}\end{matrix} \middle| \begin{matrix}\gamma\,[21]\,[21]\\ 1\quad \tilde{m}'\end{matrix} \right\rangle_q \tag{3.13}$$

where $\varepsilon([\lambda_1][\lambda_2][\lambda])$ are phases and have been given in [12]. Equation (3.13) is derived by applying g_2 to (3.7).

Now we only need four independent relations among the a_is. Let

$$y = a_3/a_2. \tag{3.14}$$

Table 6. $[\lambda]=[311]$

	123 4 5	124 3 5	134 2 5	125 3 4	135 2 4	145 2 3
1, 234 5	$\frac{1}{[3]^2}$	$\frac{[4]}{[2][3]^2}$	$\frac{[4]}{[3]!}$			
1, 235 4	$-\frac{[4]}{[2]^3[3]^2}$	$\frac{1}{[2]^6[3]^2}$	$\frac{1}{[3][2]^6}$	$\frac{[5][3]}{[2]^6}$	$\frac{[5][3]^2}{[2]^6}$	
1, 245 3		$-\frac{[3]}{[2]^4[4]}$	$\frac{1}{[4][2]^4}$	$-\frac{[5]}{[4][2]^4}$	$\frac{[5]}{[4]![2]^3}$	$\frac{[5]}{[3]!}$
23 1, 4 5	$\frac{[5]}{[3]!}$	$-\frac{[5]}{[4]![2]^3}$	$-\frac{[5]}{[4][2]^4}$	$\frac{1}{[4][2]^4}$	$\frac{[3]}{[4][2]^4}$	
24 1, 3 5		$-\frac{[5][3]^2}{[2]^6}$	$-\frac{[5][3]}{[2]^6}$	$-\frac{1}{[3][2]^6}$	$\frac{1}{[2]^6[3]^2}$	$\frac{[4]}{[2]^3[3]^2}$
25 1, 3 4				$\frac{[4]}{[3]!}$	$-\frac{[4]}{[2][3]^2}$	$\frac{1}{[3]^2}$
12, 34 5	$\frac{1}{[3]}$	$\frac{[4]}{[3]!}$				
12, 35 4	$-\frac{1}{[3]^2}$	$\frac{[2]}{[3]^2[4]}$		$\frac{[5][2]}{[4][3]}$		
3 12, 4 5	$\frac{[5]}{[3]^2}$	$-\frac{[5][2]}{[4][3]^2}$		$\frac{[2]}{[3][4]}$		
1 34 2' 5			$\frac{[5][2]}{[4][3]}$		$-\frac{[2]}{[4][3]^2}$	$-\frac{1}{[3]^2}$
1 35 2' 4					$\frac{[4]}{[3]!}$	$\frac{1}{[3]}$
1 2', 345				$\frac{[2]}{[3][4]}$	$\frac{[5][2]}{[4][3]^2}$	$\frac{[5]}{[3]^2}$

Table 7. $[\lambda]=[321]$

$\mu=1\downarrow$	124 35 6	125 34 6	124 36 5	125 36 4	126 34 5	126 35 4
$[21]_\mu[3]$	$\frac{1}{[2]^3[4]}$	$\frac{[3]}{[2]^3[4]}$	$\frac{[3]}{[2]^3[4]}$	$\frac{[5]}{[2]^3[4]}$	$\frac{[5]}{[2]^3[4]}$	$\frac{[3][5]}{[2]^3[4]}$
$[21]_\mu[21]_1\alpha$	$\frac{[3]'^2[5]}{2[5]'[4][2]^3}$	$\frac{[3]'^2[5][3]}{2[5]'[4][2]^3}$	$\frac{[3][5]}{2[5]'[4][2]^3}$	$\frac{[5]^2}{2[5]'[4][2]^3}$	$-\frac{[5]'}{2[4][2]^3}$	$-\frac{[5]'[3]}{2[4][2]^3}$
$[21]_\mu[21]_2\alpha$	$\frac{[3][5]}{2[5]'[4][2]^3}$	$-\frac{[5]}{2[5]'[4][2]^3}$	$\frac{[5]'[5]}{2[4][2]^3}$	$-\frac{[3][5]'}{2\ [4][2]^3}$	$\frac{[3]^3[5]'}{2[4][3]'^2[2]^3}$	$-\frac{[3]^2[5]'}{2[2]^3[4][3]'^2}$
$[21]_\mu[21]_1\beta$	$\frac{[3]^2[5]'}{2[2]^3[4][3]'^2}$	$\frac{[3]^3[5]'}{2[2]^3[4][3]'^2}$	$-\frac{[3][5]'}{2[4][2]^3}$	$-\frac{[5][5]'}{2[4][2]^3}$	$\frac{[5]}{2[5]'[4][2]^3}$	$\frac{[3][5]}{2[5]'[4][2]^3}$
$[21]_\mu[21]_2\beta$	$-\frac{[3][5]'}{2[4][2]^3}$	$\frac{[5]'}{2[4][2]^3}$	$-\frac{[5]^2}{2[5]'[4][2]^3}$	$\frac{[3][5]}{2[5]'[4][2]^3}$	$\frac{[3]'^2[5][3]}{2[5]'[4][2]^3}$	$-\frac{[3]'^2[5]}{2[5]'[4][2]^3}$
$[21]_\mu[1^3]$	$\frac{[3][5]}{[2]^3[4]}$	$-\frac{[5]}{[2]^3[4]}$	$-\frac{[5]}{[2]^3[4]}$	$\frac{[3]}{[2]^3[4]}$	$\frac{[3]}{[2]^3[4]}$	$-\frac{1}{[2]^3[4]}$
$\mu=2\uparrow$	134 25 6	135 24 6	134 26 5	135 26 4	136 24 5	136 25 4

Using (3.9*a*) and the orthogonality relation for the SDCs of the first and third rows in (3.8),

$$a_1a_4=-a_2a_3[3]/[5]^{1/2} \tag{3.15}$$

we get the quadratic equation for y,

$$y^2[5]+2[3]y+1=0 \tag{3.16}$$

with the solution

$$y=-([3]\pm[2])/[5]. \tag{3.17}$$

We find the only solution for y is

$$y=-([3]+[2])/[5]=-[5]'/[5] \tag{3.18}$$

where, for a given x,

$$[x]'=(q^{x/2}-q^{-x/2})/(q^{1/2}-q^{-1/2}). \tag{3.19}$$

Hence, from (3.9), (3.14) and (3.15), we obtain

$$a_3=-a_2[5]'/[5] \qquad a_1=a_2[3]'/[5]^{1/2} \qquad a_4=a_2[3][5]'/[3]'[5]. \tag{3.20}$$

Finally, using (3.10), we get

$$a_1=([3]'^2[5]/2[5]'[4][2]^3)^{1/2} \qquad a_2=([5]^2/2[5]'[4][2]^3)^{1/2}$$
$$a_3=-([5]'/2[4][2]^3)^{1/2} \qquad a_4=-([3]^2[5]'/2[4][2]^3[3]'^2)^{1/2}. \tag{3.21}$$

The final results of the SDCs are given in table 7.

4. SDCS

In this section, we will present some SDCs of $H_n(q)$ derived by using the linear equation method outlined in the above section. In the following, we will discuss a number of special cases first.

(*a*) $[\lambda_2]=1$:

$$\left\langle \begin{matrix} [\lambda] \\ m \end{matrix} \middle| [\lambda], \begin{matrix} [\lambda_1]\,[1] \\ m_1 \end{matrix} \right\rangle_q = \delta_{m_1(m)_1} \tag{4.1}$$

where $(m)_1$ is the Young tableau resulting from deleting the box with the number n in $Y_m^{[\lambda]}$.

(*b*) $[\lambda_2]=[2]$ or $[11]$: (i) suppose that i and $i+1$ are either in the same row or in the same column of the Young tableau $Y_m^{[\lambda]}$, then

$$\langle [\lambda]m|[\lambda_1]m_1[\lambda_2]\rangle_q = \delta_{m_1(m)_1} \tag{4.2}$$

where $(m)_1$ is the Young tableau resulting from deleting boxes with the numbers n and $n+1$ in $Y_m^{[\lambda]}$, (ii) when n and $n+1$ are neither in the same row nor in the same column, we can obtain a relation between the two SDCs from (3.1). The results are given in table 1.

(*c*) When both $[\lambda_1]$, $[\lambda_2]$, and $[\lambda]$ are symmetric or antisymmetric, there is only one term in the expansion. The SDCs for such cases are trivial:

$$\langle [n]|[n];[n_1][n_2]\rangle_q = 1 \tag{4.3a}$$

$$\langle [1^n]|[1^n];[1^{n_1}][1^{n_2}]\rangle_q = 1. \tag{4.3b}$$

Other non-trivial SDCs can be derived by using the linear equation method outlined in section 3. The phase convention for the SDCs is the same as that chosen for symmetric groups [12]:

$$\left\langle \begin{matrix} [\lambda] \\ m \end{matrix} \middle| [\lambda], \begin{matrix} [\lambda_1]\,[\lambda_2] \\ m_1 \;\; m_2 \end{matrix} \right\rangle_q |_{m=\min} \rangle 0 \tag{4.4}$$

where $m=\min$ means taking the index m as small as possible (the maximum Yamanouchi symbol $r_n r_{n-1}\ldots r_2 r_1$ corresponds to the smallest index $m=1$).

In tables 1–7 we list the SDCs of $H_n(q)$ for $n\leqslant 5$ and one multiplicity case for $n=6$. In the tables, (*a*) it is assumed that the index m is smaller than the index m', while $d>0$, and (*b*) the entries are the squares of the SDCs; a minus sign indicates a negative SDC.

5. Discussion

In this paper, the non-standard basis for Hecke algebras of A_{n-1} type has been discussed, and the SDCs of the Hecke algebras $H_n(q)$ for $n\leqslant 5$ and one multiplicity case for $n=6$ have been derived by using the linear equation method. This method provides us with a useful tool for deriving analytical expressions of some SDCs and IDCs of Hecke algebras. Using the Schur–Weyl duality relation between Hecke algebras and the quantum group $U_q(N)$ [11], we know that the SDCs of Hecke algebras are very useful in two aspects. First, the Hecke algebra SDCs are also special SDCs for the quantum group chain $U_q(m+n)\supset U_q(m)\oplus U_q(n)$. The general quantum group SDCs can be derived from them. Secondly, the Racah coefficients of the quantum group $U_q(N)$ can also be expressed in terms of them [12]:

$$U(\lambda_1\lambda_2\lambda\lambda_3;\lambda_{12}\lambda_{23})^{\tau_{12}\tau}_{\tau_{23}\tau'}=\sum_{m_{12}m_{23}m}\left\langle\begin{matrix}[\lambda]\\ m\end{matrix}\middle|[\lambda],\begin{matrix}\tau\,[\lambda_{12}]\,[\lambda_3]\\ m_{12}\ \ m_3\end{matrix}\right\rangle_q\left\langle\begin{matrix}[\lambda_{12}]\\ m_{12}\end{matrix}\middle|[\lambda_{12}],\begin{matrix}\tau_{12}\,[\lambda_1]\,[\lambda_2]\\ m_1\ \ m_2\end{matrix}\right\rangle_q$$

$$\times\left\langle\begin{matrix}[\lambda]\\ m\end{matrix}\middle|[\lambda],\begin{matrix}\tau'\,[\lambda_1]\,[\lambda_{23}]\\ m_1\ \ m_{23}\end{matrix}\right\rangle_q\left\langle\begin{matrix}[\lambda_{23}]\\ m_{23}\end{matrix}\middle|[\lambda_{23}],\begin{matrix}\tau_{23}\,[\lambda_2]\,[\lambda_3]\\ m_2\ \ m_3\end{matrix}\right\rangle_q \qquad (5.1)$$

where the summation is carried out under fixed m_1, m_2 and m_3. The Schur–Weyl duality between the Hecke algebras and quantum group $U_q(N)$ also enables us to obtain CGCs of $SU_q(N)$ from IDCs of Hecke algebras. Work in this direction is in progress.

References

[1] Jones V F R 1987 *Ann. Math.* **126** 335
[2] Wenzl H 1988 *Invet. Math.* **92** 349
[3] Wenzl H 1990 *Commun. Math. Phys.* **133** 383
[4] Jimbo M 1985 *Lett. Math. Phys.* **10** 63; 1986 *Lett. Math. Phys.* **11** 247
[5] Birman J and Wenzl H 1989 *Trans. Am. Math. Soc.* **313** 243
[6] Wadati M, Deguchi T and Akutsu Y 1989 *Phys. Rep.* **180** 248
[7] Zhang R B, Gould M D and Braken A J 1991 *Nucl. Phys.* B **354** 265
[8] Saleur H and Altschuler D 1991 *Nucl. Phys.* B **354** 259
[9] Dipper H and James G 1986 *Proc. London Math. Soc.* **52** 20; 1987 *Proc. London Math. Soc.* **54** 57
[10] Jones V F R 1985 *Bull. Am. Math. Soc.* **12** 103
[11] Goodman F M and Wenzl H 1990 *Adv. Math.* **82** 244
Lusztig G 1990 *J. Alg.* **131** 466
Wenzl H 1990 *Pacific J. Math.* **145** 153
[12] Chen Jin-Quan 1989 *Group Representation Theory for Physicists* (Singapore: World Scientific)
[13] Hamermesh M 1962 *Group Theory and its Applications to Physical Problems* (Massachusetts: Addison-Wesley)